J. Inverse Ill-Posed Probl. **21** (2013), 159–175
DOI 10.1515/jip-2012-0021

Certain problems of synchronization theory

Serikbai A. Aisagaliev and Maxat N. Kalimoldayev

Abstract. We investigate the global asymptotical stability of dynamical systems with cylindrical phase space, which appear in the synchronization problem of various control systems. New effective criteria of the global asymptotic stability of a countable equilibrium position are proposed. The technique of improper integrals evaluation along solutions is used.

Keywords. Global asymptotic stability, synchronization, dynamical systems, cylindrical phase space.

2010 Mathematics Subject Classification. 93C15, 93D20.

1 Introduction and Preliminaries

We shall consider the following dynamical system:

$$\frac{d\sigma}{dt} = \eta, \quad \frac{d\eta}{dt} = -g(\eta,\sigma) + z^* Cf(\sigma) - \varphi(\sigma), \quad t \in I = [0,\infty), \tag{1.1}$$

$$\dot{z} = Az + Bf(\sigma)\eta, \quad \eta(0) = \eta_0, \quad \sigma(0) = \sigma_0, \quad z(0) = z_0, \tag{1.2}$$

where A, B, C are constant $n \times n$, $n \times 1$, $n \times 1$ matrices, respectively, $f(\sigma)$, $g(\eta,\sigma)$, $\varphi(\sigma)$ are continuously differentiable functions satisfying the following conditions:

$$\begin{aligned} f(\sigma) = f(\sigma+\Delta), \quad \varphi(\sigma) = \varphi(\sigma+\Delta), \\ g(\eta,\sigma) = g(\eta,\sigma+\Delta), \quad \forall\sigma,\ \sigma \in \mathbb{R}^1, \end{aligned} \tag{1.3}$$

$$\mu_1 \le \frac{g(\eta,\sigma)}{\eta} \le \mu_2, \quad \forall\eta,\ \eta \in \mathbb{R}^1,\ \sigma \in \mathbb{R}^1,\ \mu_1 \ge 0,\ g(0,\sigma) = 0, \tag{1.4}$$

where Δ is a period and derivatives $\frac{\partial f}{\partial\sigma}, \frac{\partial g}{\partial\sigma}, \frac{\partial g}{\partial\eta}, 0 \le \sigma \le \Delta, \forall\eta \in \mathbb{R}^1$ are bounded functions.

We assume that systems (1.1)–(1.2) satisfy local conditions of an existence theorem for solutions and all solutions of these systems are continuable on $\mathbb{R}$.

It is known that equations (1.1)–(1.4) are very popular in theory of synchronization, and they have numerous applications.

We assume that equilibrium position of system (1.1)–(1.2) form a countable set

$$\Lambda = \{\eta_* = 0,\, \sigma = \sigma_*,\, z_* = 0 \mid \varphi(\sigma_*) = 0\}.$$

It is said that the system (1.1)–(1.2) *is globally asymptotically stable* if each solution $(\eta(t), \sigma(t), z(t))$ of the system tends to a point from the set Λ as $t \to \infty$

Problem 1.1. Find a criterion of the global asymptotical stability of the system (1.1)–(1.2) if nonlinear functions $f(\sigma), \varphi(\sigma), g(\eta, \sigma)$ satisfy the conditions (1.3)–(1.4), and

$$\int_{\sigma}^{\sigma+\Delta} \varphi(\xi) d\xi = 0, \quad \forall \sigma,\ \sigma \in \mathbb{R}^1. \tag{1.5}$$

Problem 1.2. Find a criterion of the global asymptotical stability of the system (1.1)–(1.2) with nonlinear functions $f(\sigma)$, $\varphi(\sigma)$, $g(\eta, \sigma)$, which satisfy conditions (1.3)–(1.4), and

$$\int_{\sigma}^{\sigma+\Delta} \varphi(\xi) d\xi = \alpha, \quad \alpha \neq 0, \quad \forall \sigma,\ \sigma \in \mathbb{R}^1. \tag{1.6}$$

The class of differential equations (1.1)–(1.2) describes dynamical systems [1,8,9], and also dynamics of Buass–Sard's regulator [6]. Frequency conditions of the global asymptotic stability for system (1.1)–(1.2) and conditions of existence of limit cycles and circular solutions in such systems have been investigated in [7]. The review of scientific literature on dynamics of phase systems can be found in [4]. The present paper is a continuation of the study, which have been started in papers [2, 3, 5].

Let us note that it is a difficult task to apply the frequency criteria for all values of the parameter ω, which changes from $-\infty$ up to $+\infty$. Therefore, development of new research methods for global asymptotic stability of such systems is actual, i.e. in this paper improper integrals evaluation is one of the new results that allow to obtain conditions of the global asymptotic stability without using a Lyapunov function and the frequency theorem.

We shall consider equations (1.2):

$$\dot{z} = Az + Bf(\sigma), \quad z(0) = z_0, \quad t \in I = [0, \infty). \tag{1.7}$$

The characteristic polynom of the matrix A has the following form:

$$\Delta(\lambda) = \det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

According to Cayley–Hamilton's theorem, $\Delta(A) = 0$. That is why

$$A^n = -a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_1 A - a_0 I_n,$$

where I_n is the unit $n \times n$ matrix.

Lemma 1.3. *We assume that there is a constant vector $\theta^* \in \mathbb{R}^n$ such that*

$$\theta B = 0, \theta AB = 0, \dots, \theta A^{n-2} B = 0, \quad k = \theta A^{n-1} B \neq 0, \tag{1.8}$$

where $(*)$ *is a sign of transposition, and* $\theta = (\theta_1, \dots, \theta_n)$ *is a row vector. Then equation* (1.7) *can be represented as*

$$\begin{aligned} &\dot{y} = y_2, \dot{y}_2 = y_3, \dots, \dot{y}_{n-1} = y_n, \\ &\dot{y}_n = -a_0 y_1 - a_1 y_2 - \dots - a_{n-1} y_n + k f(\sigma)\eta, \end{aligned} \tag{1.9}$$

where $\theta z = y_1$, $\theta A z = y_2, \dots, \theta A^{n-2} z = y_n$, $z = z(t)$, $y_i = y_i(t)$, $i = \overline{1, n}$.

Proof. Multiplying the left side of (1.7) by θ, we obtain that

$$\theta \dot{z} = \theta A z + \theta B f(\sigma)\eta = \theta A z,$$

where $\theta B = 0$ by the condition (1.8) of the lemma. Hence, taking into account that $y_1 = \theta z$, $y_2 = \theta A z$, we obtain $\dot{y}_1 = y_2$.

Since $\theta \dot{z} = \theta A z$, we have that

$$\theta \ddot{z} = \theta A \dot{z} = \dot{y}_2 = \theta A[Az + Bf(\sigma)\eta] = \theta A^2 z + \theta AB f(\sigma)\eta = \theta A^2 z,$$

where $\theta AB = 0$ by condition (1.8).

Hence $\dot{y}_2 = y_3$, where $y_3 = \theta A^2 z$, etc. Finally, we have that $\dot{y}_{n-1} = y_n$, where $y_{n-1} = \theta A^{n-2} z$, $y_n = \theta A^{n-1} z$. Next,

$$\dot{y}_n = \theta A^{n-1} \dot{z} = \theta A^n z + \theta A^{n-1} B f(\sigma)\eta = \theta A^n z + k f\,(\sigma)\,\eta,$$

where $k = \theta A^{n-1} B \neq 0$.

Since

$$\begin{aligned} \theta A^n z &= \theta(-a_{n-1} A^{n-1} - a_{n-2} A^{n-2} - \dots - a_1 A - a_0 I_n) z \\ &= -a_0 y_1 - a_1 y_2 - \dots - a_{n-1} y_n, \end{aligned}$$

one can obtain that

$$\dot{y}_n = -a_0 y_1 - a_1 y_2 - \dots - a_{n-1} y_n + k f(\sigma)\eta.$$

Thus, conditions (1.8) imply that system (1.7) can be transformed to system (1.9). The lemma is proved. □

Lemma 1.4. *Let all conditions of Lemma* 1.3 *hold, and the rank of the* $n \times n$ *matrix*

$$R = \|\theta^*, A^*\theta^*, \dots, A^{*n-1}\theta^*\| \tag{1.10}$$

be equal to n. *Then:*

(i) $y = R^* z$ *and* $z = R^{*-1} y$, *where* $y = y(t) = (y_1(t), \dots, y_n(t))$, $t \in I$, *and* $z = z(t) = (z_1(t), \dots, z_n(t))$, *are solutions of differential equations* (1.9) *and* (1.7) *respectively,*

(ii) *if* $\lim_{t\to\infty} y(t) = 0$, *then* $\lim_{t\to\infty} z(t) = 0$.

Proof. Since $y_1 = \theta z, y_2 = \theta A z, \dots, y_{n-1} = \theta A^{n-2} z, y_n = \theta A^{n-1} z$, we have that

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \theta z \\ \theta A z \\ \vdots \\ \theta A^{n-1} z \end{pmatrix} = \begin{pmatrix} \theta \\ \theta A \\ \vdots \\ \theta A^{n-1} \end{pmatrix} z = R^* z.$$

By the conditions of the last lemma, the matrix R is non-special. Hence, there exist inverse matrices R^{-1}, R^{*-1}, and $z = R^{*-1} y$. On the other hand, according to (1.10), the pair (θ^*, A^*) is controllable. By controllability of (θ^*, A^*), the equalities $\theta z = 0$, $\theta A z = 0, \dots, \theta A^{n-1} z = 0$ imply $z = 0$. Hence, $\lim_{t\to\infty} y(t) = 0$ implies that $\lim_{t\to\infty} z(t) = 0$. The lemma is proved. □

According to Lemmas 1.3 and 1.4, if equalities (1.8) hold and the rank of matrix R is equal to n, then system (1.9) is equivalent to system (1.7). Moreover, according to $\lim_{t\to\infty} y(t) = 0$, it follows that $\lim_{t\to\infty} z(t) = 0$.

2 Properties of solutions

Let us consider the differential equation (1.9).

Lemma 2.1. *We assume that the conditions of Lemmas* 1.3 *and* 1.4 *hold and that* $k \neq 0$. *Then along the solution of system* (1.9) *the following identity is true:*

$$f(\sigma(t))\eta(t) = k^{-1}[\omega(t) + a_0 y_1(t) + a_1 y_2(t) + \dots + a_{n-1} y_n(t)], \quad t \in I, \quad (2.1)$$

where $\omega(t) = \dot{y}_n(t)$, $t \in I$.

Proof. It can be easily seen that along a solution of (1.9) the following identities are true:

$$\dot{y}_i(t) = y_{i+1}(t), \quad i = \overline{1, n-1},$$
$$\dot{y}_n(t) = \omega(t) = -a_0 y_1(t) - \dots - a_{n-1} y_n(t) + k f(\sigma(t))\eta(t), \quad t \in I,$$

where $k = \theta A^{n-1} B \neq 0$. Equation (2.1) follows from the last identity. The lemma is proved. □

3 Improper integrals

We suppose that all conditions of Lemmas 1.3 and 1.4 hold. Now equations of the system (1.1)–(1.2) can be written as

$$\frac{d\sigma}{dt} = \eta, \dot{\eta} = -g(\eta,\sigma) + y^* R^{-1} C f(\sigma) - \varphi(\sigma), \quad t \in I = [0,\infty), \tag{3.1}$$

$$\begin{gathered} \dot{y}_1 = y_2, \dot{y}_2 = y_3, \dots, \dot{y}_{n-1} = y_n, \\ \dot{y}_n = -a_0 y_1 - a_1 y_2 - \dots - a_{n-1} y_n + k f(\sigma)\eta, \quad t \in I, \end{gathered} \tag{3.2}$$

where $z = R^{*-1} y$, $z^* = y^* R^{-1}$.

Theorem 3.1. *Let the conditions of Lemmas* 1.3 *and* 1.4 *hold. Then along a solution of the system* (3.1)–(3.2) *the following equality holds:*

$$\begin{aligned} &\lim_{T\to\infty} \int_0^T [g(\eta,\sigma)\eta + \varphi(\sigma)\dot{\sigma} - y^*(t) R^{-1} C f(\sigma)\eta]dt \\ &\qquad = \lim_{T\to\infty} \left[-\frac{1}{2}\eta^2(T) + \frac{1}{2}\eta^2(0) \right], \end{aligned} \tag{3.3}$$

where $\eta = \eta(t)$, $\sigma = \sigma(t)$, $y = y(t)$, $\dot{\sigma}(t) = \eta(t)$, $t \in I$.

Proof. Multiply the second equation of (3.1) by $\eta = \eta(t)$, and obtain that

$$\eta\dot{\eta} = -\,g(\eta,\sigma)\eta + y^* R^{-1} C f(\sigma)\eta - \varphi(\sigma)\eta, \quad \eta = \dot{\sigma}.$$

Integrating by t the given identity from 0 up to T, $T \to \infty$, and multiplying by $\delta_1 \geq 0$, we obtain equality (3.3). The theorem is proved. □

Theorem 3.2. *We assume that the conditions of Lemmas* 1.3–2.1 *are true. Then it is true that*

$$\begin{aligned} I_2 &= \lim_{T\to\infty} \int_0^T y^* R^{-1} C f(\sigma(t))\eta(t) dt \\ &= \lim_{T\to\infty} \int_0^T [M_1 y_1^2(t) + \dots + M_n y_n^2(t)] dt \\ &\qquad + \lim_{T\to\infty} y^*(T) N y(T) - y^*(0) N y(0), \end{aligned} \tag{3.4}$$

along a solution of system (3.1)–(3.2)*, where* $M_1, \dots, M_n$*, and the* $n \times n$ *constant matrix* N *are determined by the parameters* $R, C, a_0, \dots, a_{n-1}, k$ *– of the system* (3.1)–(3.2).

Proof. Let

$$a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \in \mathbb{R}^n, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

Then

$$y^* R^{-1} C f(\sigma)\eta = y^* R^{-1} C [k^{-1}(\omega + a^* y)],$$

where $y = y(t)$, $f(\sigma) = f(\sigma(t))$, $\eta = \eta(t)$ and $\omega = \omega(t)$, $t \in I$, and the product $f(\sigma(t))\eta(t)$, $t \in I$, is defined by formula (2.1). Hence,

$$I_2 = \lim_{T\to\infty} \int_0^T y^* R^{-1} C f(\sigma(t))\eta(t) dt = I_{21} + I_{22}, \tag{3.5}$$

where

$$\begin{aligned} I_{21} &= \lim_{T\to\infty} \int_0^T y^*(t) R^{-1} C k^{-1} \omega(t) dt, \\ I_{22} &= \lim_{T\to\infty} \int_0^T y^*(t) R^{-1} C k^{-1} a^* y(t) dt. \end{aligned} \tag{3.6}$$

Let us calculate the integral I_{21}. Let

$$\overline{C} = R^{-1} C k^{-1} = \begin{pmatrix} \overline{C_1} \\ \vdots \\ \overline{C_n} \end{pmatrix} \in \mathbb{R}^n.$$

Then $I_{21} = \lim_{T\to\infty} \int_0^T \sum_{j=1}^n y_j(t)\overline{C_j}\dot{y}_n(t)dt$. Note that

$$\begin{aligned}
j = n: \quad & \int_0^T y_n \dot{y}_n dt = \int_0^T \frac{d}{dt}\left(\frac{1}{2}y_n^2\right) dt = \frac{1}{2}y_n(T) - \frac{1}{2}y_n(0), \\
j = n-1: \quad & \int_0^T y_{n-1}\dot{y}_n dt = \int_0^T \left[\frac{d}{dt}(y_n y_{n-1}) - \dot{y}_{n-1} y_n\right] dt \\
& \qquad = \int_0^T \frac{d}{dt}(y_{n-1} y_n) dt - \int_0^T y_n^2(t) dt, \\
& \dot{y}_{n-1} = y_n, \\
j = n-2: \quad & \int_0^T y_{n-2}\dot{y}_n(t) dt = \int_0^T \frac{d}{dt}\left(y_{n-2} y_n - \frac{1}{2}y_{n-1}^2\right) dt
\end{aligned}$$

and

$$j = n-3: \qquad \int_0^T y_{n-3}\dot{y}_n dt = \int_0^T \frac{d}{dt}\left(y_{n-3}y_n - y_{n-2}y_{n-1}\right) dt + \int_0^T y_{n-1}^2(t)\, dt$$

and so on.

Finally, we obtain that (see also (3.6))

$$\begin{aligned} I_{21} &= \lim_{T\to\infty}\int_0^T y^*(t)R^{-1}Ck^{-1}\omega(t)dt \\ &= \lim_{T\to\infty}\int_0^T [\Gamma_1 y_1^2 + \cdots + \Gamma_n y_n^2]dt \\ &\quad + \lim_{T\to\infty} y^*(T)P_0 y(T) - y^*(0)P_0 y(0), \end{aligned} \tag{3.7}$$

where $\Gamma_1, \ldots, \Gamma_2$, the $n \times n$ matrix P_0 are defined by the vector $\overline{C} \in \mathbb{R}^n$.

Let us calculate the integral I_{22}. Set $D = R^{-1}Ck^{-1}a^* = \|D_{ij}\|, i, j = \overline{1,n}$. Then

$$I_{22} = \lim_{T\to\infty}\int_0^T y^*(t)Dy(t)dt = \lim_{T\to\infty}\int_0^T \sum_{i=1}^n\sum_{j=1}^n y_1(t)D_{ij}y_j(t)dt.$$

It is easy to verify that

$$\begin{aligned} \int_0^T y_1y_2dt &= \int_0^T y_1\dot{y}_1dt = \int_0^T \frac{d}{dt}\left(\frac{1}{2}y_1^2\right)dt = \frac{1}{2}y_1^2(T) - \frac{1}{2}y_1^2(0), \\ \int_0^T y_1y_3dt &= \int_0^T (y_1y_2)dt - \int_0^T y_2^2dt, \\ \int_0^T y_1y_4dt &= \int_0^T \frac{d}{dt}\left(y_1y_3 - \frac{1}{2}y_2^2\right)dt, \end{aligned}$$

and so on.

Now, the improper integral I_{22} can be represented as

$$\begin{aligned} I_{22} &= \lim_{T\to\infty}\int_0^T y^*(t)R^{-1}Ck^{-1}a^*y(t)dt \\ &= \lim_{T\to\infty}\int_0^T \left[L_1y_1^2 + \cdots + L_ny_n^2\right]dt \\ &\quad + \lim_{T\to\infty} y^*(T)P_1y(T) - y^*(0)P_1y(0), \end{aligned} \tag{3.8}$$

where the numbers $L_1, \dots, L_n$ and the matrix P_1 are defined by the matrix D. By (3.5)–(3.8) we have

$$
\begin{aligned}
I_2 &= \lim_{T\to\infty}\int_0^T y^*(t)R^{-1}Cf(\sigma(t))\eta(t)dt \\
&= \lim_{T\to\infty}\int_0^T [(\Gamma_1 + L_1)y_1^2(t) + \dots + (\Gamma_n + L_1)y_n^2(t)]dt \\
&\qquad + \lim_{T\to\infty} y^*(T)(P_0 + P_1)y(T) - y^*(0)(P_0 + P_1)y(0). \qquad \square
\end{aligned}
$$

Theorem 3.3. *We suppose that the conditions of Lemmas* 1.3–2.1 *hold. Then along a solution of the system* (3.1)–(3.2) *the improper integral is equal to*

$$
\begin{aligned}
I_3 &= \lim_{T\to\infty}\int_0^T g(\eta(t),\sigma(t))\eta(t)dt \\
&\qquad - \lim_{T\to\infty}\int_0^T [M_1y_1^2(t) + \dots + M_ny_n^2(t)]dt \\
&= -\frac{1}{2}\lim_{T\to\infty}\eta^2(T) + \lim_{T\to\infty} y^*(T)Ny(T) \\
&\qquad - \lim_{T\to\infty}\int_{\sigma(0)}^{\sigma(T)}\varphi(\xi)d\xi + \frac{1}{2}\eta^2(0) - y^*(0)Ny(0).
\end{aligned} \tag{3.9}
$$

Proof. From equality (3.3), taking into account (3.4), and also

$$
\lim_{T\to\infty}\int_0^T \varphi(\sigma(t))\dot\sigma(t)dt = \lim_{T\to\infty}\int_{\sigma(0)}^{\sigma(T)}\varphi(\xi)d\xi,
$$

we obtain that

$$
\begin{aligned}
&\lim_{T\to\infty}\int_0^T g(\eta(t),\sigma(t))\eta(t)dt - \lim_{T\to\infty}\int_0^T [M_1y_1^2(t) + \dots + M_ny_n^2(t)]dt \\
&\qquad - \lim_{T\to\infty} y^*(T)Ny(T) + y^*(0)Ny(0) \\
&\qquad + \lim_{T\to\infty}\int_{\sigma(0)}^{\sigma(T)}\varphi(\xi)d\xi \\
&= \lim_{T\to\infty}\left[-\frac{1}{2}\eta^2(T)\right] + \frac{1}{2}\eta^2(0).
\end{aligned}
$$

According to this, the ratio (3.9) follows. The theorem is proved. $\square$

4 Global asymptotic stability

We shall consider the solution of Problem 1.1 in this section.

Theorem 4.1. *Let the following conditions hold:*

(i) *there is a vector* $\theta^* \in \mathbb{R}^n$ *such that* $\theta B = 0$, $\theta AB = 0, \ldots, \theta A^{n-2}B = 0$, $k = \theta A^{n-1}B \neq 0$,

(ii) *the variable* $\mu_1 \geq 0$,

(iii) *the rank of a matrix* $R = \|\theta^*, A^*\theta^*, \ldots, A^{*n-1}\theta^*\|$ *is equal n,*

(iv) $\int_\sigma^{\sigma+\Delta} \varphi(\xi)d\xi = 0$,

(v) $M_i < 0$, $i = \overline{1,n}$, *the matrix* $N \leq 0$.

Then the set of equilibria $\wedge$ *of the system* (1.1)–(1.2) *is globally asymptotically stable.*

Proof. According to restrictions (1.4), the following inequalities are true:

$$\mu_1\eta^2 \leq g(\eta,\sigma)\eta \leq \mu_2\eta^2, \quad \forall\sigma \in \mathbb{R}^1,\ \forall\eta \in \mathbb{R}^1.$$

Since $\mu_1 \geq 0$, the product $g(\eta,\sigma)\eta \geq 0$, $\forall\sigma$, $\sigma \in \mathbb{R}^1$, $\forall\eta$, $\eta \in \mathbb{R}^1$. Since

$$\int_\sigma^{\sigma+\Delta} \varphi(\xi)d\xi = 0,$$

it follows that

$$\left|\lim_{T\to\infty}\int_{\sigma(0)}^{\sigma(T)} \varphi(\xi)d\xi\right| < \infty.$$

According to (3.9) and by virtue of $g(\eta(t),\sigma(t))\eta(t) \geq 0$, $\forall t$, $t \in I$. It holds that

$$\begin{aligned}
&\lim_{T\to\infty}\int_0^T g(\eta(t),\sigma(t))\eta(t)dt \\
&\quad = \lim_{T\to\infty}\int_0^T \sum_{i=2}^n M_i y_i^2(t)dt - \frac{1}{2}\lim_{T\to\infty}\eta^2(T) \\
&\qquad + \lim_{T\to\infty} y^*(T)Ny(T) - \lim_{T\to\infty}\int_{\sigma(0)}^{\sigma(T)} \varphi(\xi)d\xi \\
&\qquad + \frac{1}{2}\eta^2(0) - y^*(0)Ny(0) \\
&\quad \geq 0.
\end{aligned} \tag{4.1}$$

Since $M_i < 0$, $i = \overline{1,n}$, the matrix $N \le 0$, the inequality (4.1) holds if and only if functions $y_i(t)$, $t \in I$, $\eta(t)$, $t \in I$, are bounded. Indeed, if $y_i(t)$, $t \in I$, $\eta(t)$, $t \in I$, are not bounded, then

$$0 \le \lim_{T\to\infty} \int_0^T g(\eta(t), \sigma(t))\eta(t)dt \le -\infty.$$

It is impossible. So, it is proved, that $\|y(t)\| < \infty$, $|\eta(t)| < \infty$, $\forall t, t \in I$. Then, according to (4.1), it follows that

$$0 \le \lim_{T\to\infty} \int_0^T g(\eta(t), \sigma(t))\eta(t)dt \le \infty. \tag{4.2}$$

By the formulation of the problem, the partial derivatives

$$\frac{\partial g(\eta(t), \sigma(t))}{\partial \sigma}, \quad \frac{\partial g(\eta(t), \sigma(t))}{\partial \eta}, \quad t \in I,$$

are bounded functions. Hence, the function $g(\eta(t), \sigma(t))$ is uniformly continuous in η, σ. According to boundedness of $\eta(t) = \dot{\sigma}(t)$, $t \in I$, the function $\sigma(t)$, $t \in I$, is uniformly continuous. Since $|\eta(t)| < \infty$, $\forall t, t \in I$, $g(\eta, \sigma) = g(\eta, \sigma + \Delta)$, so the function $g(\eta(t), \sigma(t))$, $t \in I$, is bounded. Then, according to (3.1), the following inequality is true:

$$|\dot{\eta}(t)| \le |g(\eta(t), \sigma(t))| + |y^*(t)R^{-1}Cf(\sigma(t))| + |\varphi(\sigma(t))| < \infty,$$

where the periodic functions $f(\sigma), \varphi(\sigma)$, $\sigma \in \mathbb{R}^1$, are bounded.

Therefore, the function $\eta(t)$, $t \in I$, is uniformly continuous. So, it is proved, that the functions $\eta(t), \sigma(t)$, $t \in I$, are uniformly continuous. According to (4.2) it follows that

$$\lim_{T\to\infty} g(\eta(t), \sigma(t)) = 0, \quad \lim_{T\to\infty} \eta(t) = 0. \tag{4.3}$$

Since $g(\eta, \sigma)$ is a periodic function in σ where $g(\eta, \sigma_*) = 0$, $\varphi(\sigma_*) = 0$, by (4.3) we have that $\lim_{t\to\infty} \sigma(t) = \sigma_*$.

On the other hand, equality (3.9) implies that

$$I_3 = \lim_{T\to\infty} \int_0^T g(\eta(t), \sigma(t))\eta(t)dt - \lim_{T\to\infty} \int_0^T \sum_{i=2}^n M_i y_i^2(t)dt < \infty,$$

where $M_i < 0$, $i = \overline{1,n}$, $g(\eta(t), \sigma(t))\eta(t) \ge 0$, $t \in I$. Then

$$\lim_{T\to\infty} \int_0^T \sum_{i=2}^n M_i y_i^2(t)dt < \infty. \tag{4.4}$$

Let us consider equation (3.2) where the functions $y_i(t)$, $t \in I$, $i = \overline{1,n}$, are bounded as well as $\eta(t)$, $t \in I$. Then

$$|\dot{y}_n(t)| \le |a_0||y_1(t)| + \cdots + |a_{n-1}||y_n(t)| + |k||f(\sigma(t))||\eta(t)| < \infty,\ \forall t, t \in I.$$

According to this, the function $y_n(t)$, $t \in I$, is uniformly continuous. Since

$$\dot{y}_1 = y_2, \ldots, \dot{y}_{n-1} = y_n,$$

the functions $y_i(t)$, $i = \overline{1, n-1}$, are also uniformly continuous. Then according to (4.4) we obtain that $\lim_{t\to\infty} y_i(t) = 0$, $i = \overline{1,n}$.

So we have proved that $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} \eta(t) = 0$, $\lim_{t\to\infty} \sigma(t) = \sigma_*$, where $\varphi(\sigma_*) = 0$. It means that the stationary set Δ of the system (1.1)–(1.2) is globally asymptotically stable. The theorem is proved. □

Let us consider the solution of Problem 1.2, when (see (1.6))

$$\int_{\sigma}^{\sigma+\Delta} \varphi(\xi)d\xi = \alpha, \quad \alpha \ne 0.$$

5 Improper integrals

On the basis of Lemmas 1.3–2.1 and Theorems 3.1–3.3 we can obtain estimations of improper integrals.

Theorem 5.1. *Let nonlinearity $g(\eta,\sigma)$ satisfies condition* (1.4). *Then for any numbers $\tau_1 \ge 0, \tau_2 \ge 0$ along the solution of the system* (1.1) *the improper integral satisfies*

$$I_4 = \lim_{T\to\infty} \int_0^T [(\tau_1\mu_1 - \tau_2\mu_2)\eta^2(t) + (\tau_2 - \tau_1)g(\eta(t),\sigma(t))\eta(t)]dt \le 0. \quad (5.1)$$

Proof. According to the condition (1.4) along the solution of the system (1.1), the following inequalities are true:

$$\begin{aligned} \mu_1\eta^2(t) - g(\eta(t),\sigma(t))\eta(t) &\le 0, \\ -\mu_2\eta^2(t) + g(\eta(t),\sigma(t))\eta(t) &\le 0, \quad \forall t,\ t \in I = [0,\infty). \end{aligned} \quad (5.2)$$

Let us multiply the first inequality from (5.2) by $\tau_1 \ge 0$ and the second inequality by $\tau_2 \ge 0$ and then we sum them up. As a result we obtain

$$(\tau_1\mu_1 - \tau_2\mu_2)\eta^2(t) + (\tau_2 - \tau_1)g(\eta(t),\sigma(t))\eta(t) \le 0, \quad \forall t,\ t \in I. \quad (5.3)$$

Integrating on τ in limits from 0 up to T the inequality (5.3) with the subsequent transition to the limit at $T \to \infty$, we obtain the estimation (5.1). The theorem is proved. □

Theorem 5.2. *Let numbers*

$$\alpha = \int_\sigma^\Delta \varphi(\xi)d\xi,\ \ \beta = \int_\sigma^\Delta |\varphi(\xi)|d\xi,\ \ \nu = \frac{\alpha}{\beta},\ \ \tau_3,\ \ \tau_4 = \frac{\nu^2\tau_3^2}{4\tau_5} > 0,\ \ \tau_5 > 0.$$

Then along the solution of the system (1.1) *the improper integral satisfies*

$$I_5 = \lim_{T\longrightarrow\infty}\int_0^T [\tau_3\varphi(\sigma(t))\dot\sigma(t) - \tau_4\varphi^2(\sigma(t)) - \tau_5\eta^2(t)]dt \le C_1 < \infty, \quad (5.4)$$

$C_1 = \text{const}$*, where* $\sigma = \sigma(t)$*,* $\eta = \eta(t) = \dot\sigma(t)$*,* $t \in I$*,* $|C_1| < \infty$.

Proof. We note that for the function $\Phi(\sigma) = \varphi(\sigma) - \nu|\varphi(\sigma)|$, $\sigma \in \mathbb{R}^1$, the integral

$$\int_0^\sigma \Phi(\xi)d\xi = \alpha - \nu\beta = 0 \quad (5.5)$$

by virtue of $\nu = \alpha/\beta$. Along the solution of the system (1.1) we consider the function

$$S(t) = \varphi(\sigma)\tau_3\dot\sigma - \tau_4\varphi^2(\sigma) - \tau_5\dot\sigma^2 - \tau_3\Phi(\sigma)\dot\sigma, \quad t \ge 0, \quad (5.6)$$

where $\sigma = \sigma(t)$, $\dot\sigma = \dot\sigma(t) = \eta(t)$, $\varphi(\sigma) = \varphi(\sigma(t))$, $\Phi(\sigma) = \Phi(\sigma(t))$, $t \in I$.

Let us show that $S(t) \le 0$, $\forall t$, $t \in I$. Indeed, since

$$\tau_3\varphi(\sigma)\dot\sigma - \tau_3\Phi(\sigma)\dot\sigma = \tau_3\nu|\varphi(\sigma)|\dot\sigma,$$

we get $S(t) = \nu\tau_3|\varphi(\sigma)|\dot\sigma - \tau_4\varphi^2(\sigma) - \tau_5\dot\sigma^2$, $t \in I$. From here, taking into account that

$$\nu\tau_3|\varphi(\sigma)|\dot\sigma - \tau_5\dot\sigma^{\,2} = -\left\{\left[\sqrt{\tau_5}\dot\sigma - \frac{\tau_3\nu}{2\sqrt{\tau_5}}|\varphi(\sigma)|\right]^2 - \frac{\tau_3^2\nu^2}{4\tau_5}\varphi^2(\sigma)\right\},$$

we obtain

$$\begin{aligned} S(t) &= -\left[\sqrt{\tau_5}\dot\sigma - \frac{\tau_3\nu}{2\sqrt{\tau_5}}|\varphi(\sigma)|\right]^2 + \left(\frac{\tau_3^2\nu^2}{4\tau_5} - \tau_4\right)\varphi^2(\sigma) \\ &= -\left[\sqrt{\tau_5}\dot\sigma - \frac{\tau_3\nu}{2\sqrt{\tau_5}}|\varphi(\sigma)|\right]^2, \quad t \in I, \end{aligned}$$

by virtue of the equality $\tau_3^2\nu^2 - 4\tau_4\tau_5 = 0$. Hence, $S(t) \le 0$, $t \in I$.

Then according to (5.5)

$$\varphi(\sigma)\tau_3\dot\sigma - \tau_4\varphi^2(\sigma) - \tau_5\dot\sigma^2 \le \tau_3\Phi(\sigma)\dot\sigma, \quad t \in I, \quad (5.7)$$

integrating the inequality (5.6) by t in limits from 0 up to T and taking into account (5.5), we obtain the estimation (5.4). The theorem is proved. □

Theorem 5.3. *Let the conditions of Lemmas* 1.3–2.1 *and Theorem* 5.1 *hold. Then for any numbers* $\delta_1 \geq 0$, $\delta_2 \geq 0$ *along the solution of the system* (3.1), (3.2) *the improper integral satisfies*

$$\begin{aligned} I_6 = \lim_{T\to\infty} \int_0^T &\big\{\delta_2(\tau_1\mu_1 - \tau_2\mu_2)\eta^2(t) + [\delta_1 + \delta_2(\tau_2 - \tau_1)]g(\eta(t), \sigma(t))\eta(t) \\ &+ \delta_1\varphi(\sigma(t))\eta(t) - \delta_1[M_1 y_1^2(t) + \cdots + M_n y_n^2(t)]\big\}dt \\ \leq &\left\{\lim_{T\to\infty}\left[-\frac{1}{2}\eta^2(T)\right] + \lim_{T\to\infty} y^*(T)Ny(T)\right\}\delta_1 \\ &+ \delta_1\left[\frac{1}{2}\eta^2(0) - y^*(0)Ny(0)\right], \end{aligned} \tag{5.8}$$

where $\eta(t) = \dot{\sigma}(t)$, $t \in I$.

Proof. Since the conditions of Lemmas 1.3–2.1 hold, equality (3.9) is true. From the statement of Theorem 5.1 the estimation (5.1) follows. Then the improper integral

$$\begin{aligned} I_6 &= \delta_1 I_3 + \delta_2 I_4 \\ &= \lim_{T\to\infty} \int_0^T \big\{\delta_1 g(\eta(t), \sigma(t))\eta(t) + \delta_1\varphi(\sigma(t))\eta(t) \\ &\qquad - \delta_1[M_1 y_1^2(t) + \cdots + M_n y_n^2(t)]\big\}dt \\ &\quad + \lim_{T\to\infty} \int_0^T [\delta_2(\tau_1\mu_1 - \tau_2\mu_2)\eta^2(t) + \delta_2(\tau_2 - \tau_1)g(\eta(t), \sigma(t))\eta(t)]dt \\ &\leq \delta_1\left\{\lim_{T\to\infty}\left[-\frac{1}{2}\eta^2(T)\right] + \lim_{T\to\infty} y^*(T)Ny(T)\right\} \\ &\quad + \delta_1\left[\frac{1}{2}\eta^2(0) - y^*(0)Ny(0)\right]. \end{aligned}$$

According to this the estimation (5.8) is obtained. The theorem is proved. □

Theorem 5.4. *Let the conditions of Theorem* 5.3 *hold, and* $\delta_1 > 0$, $\delta_2 > 0$, $\tau_1 \geq 0$, $\tau_2 > 0$, $\mu_1 \geq 0$, $\mu_2 > 0$, $\tau_2\mu_2 - \tau_1\mu_1 > 0$. *Then along a solution of the system* (3.1)–(3.2) *the improper integrals satisfies*

$$\begin{aligned} I_7 = \lim_{T\to\infty} \int_0^T \bigg\{&[\delta_1 + \delta_2(\tau_2 - \tau_1)]g(\eta(t), \sigma(t))\eta(t) \\ &- \delta_1[M_1 y_1^2(t) + \cdots + M_n y_n^2(t)] \\ &+ \frac{\delta_1^2\nu^2}{4\delta_2(\tau_2\mu_2 - \tau_1\mu_1)}\varphi^2(\sigma(t))\bigg\}dt \end{aligned} \tag{5.9}$$

$$\leq \lim_{T\to\infty}\left[-\frac{1}{2}\eta^2(T) + +y^*(T)Ny(T)\right]$$
$$+ \delta_1\left[\frac{1}{2}\eta^2(0) - y^*(0)Ny(0)\right] + \gamma, \quad 0 \leq \gamma < \infty, \tag{5.10}$$

where $C_1 = \lim_{T\to\infty} \delta_1 \int_0^T \Phi(\sigma)d\sigma$, $\int_0^\Delta \Phi(\sigma)d\sigma = 0$, $C_1 = I_5 + \gamma$, I_5 *is defined by formula* (5.4).

Proof. Since the conditions of Theorem 5.3 hold, the estimation (5.8) is true. From (5.8) under the conditions

$$\tau_3 = \delta_1, \quad \tau_5 = \delta_2(\tau_2\mu_2 - \tau_1\mu_1), \quad \tau_4 = \frac{\tau_3^2\nu^2}{4\tau_5} = \frac{\delta_1^2\nu^2}{4\delta_2(\tau_2\mu_2 - \tau_1\mu_1\ )} > 0$$

we obtain

$$I_6 = \lim_{T\to\infty}\int_0^T \big\{[\delta_1 + \delta_2(\tau_2 - \tau_1)]g(\eta(t),\sigma(t))\eta(t)$$
$$- \delta_1[M_1y_1^2(t) + \cdots + +M_ny_n^2(t)] + \tau_4\varphi^2(\sigma(t))\big\}dt$$
$$+ \lim_{T\to\infty}\int_0^T [\tau_3\varphi(\sigma)\dot\sigma(t) - \tau_5\dot\sigma^2(t) - \tau_4\varphi^2(\sigma(t))]dt$$
$$\leq \lim_{T\to\infty}\delta_1\left[-\frac{1}{2}\eta^2(T) + y^*(T)Ny(T)\right] + \delta_1\left[\frac{1}{2}\eta^2(0) - y^*(0)Ny(0)\right].$$

The values τ_3, $\tau_4 > 0$, $\tau_5 > 0$ are chosen in such way that the conditions of Theorem 5.2 hold. Hence,

$$I_5 = \lim_{T\to\infty}\int_0^T [\tau_3\varphi(\sigma(t))\dot\sigma(t) - \tau_5\dot\sigma^2(t) - \tau_4\varphi^2(\sigma(t))]dt \leq C_1, \quad |C_1| < \infty.$$

Then the improper integral

$$I_6 = \lim_{T\to\infty}\int_0^T \big\{[\delta_1 + \delta_2(\tau_2 - \tau_1)]g(\eta(t),\sigma(t))\eta(t)$$
$$- \delta_1[M_1y_1^2(t) + \cdots + M_ny_n^2(t)] + \tau_4\varphi^2(\sigma(t))\big\}dt + I_5$$
$$\leq \lim_{T\to\infty}\delta_1\left[-\frac{1}{2}\eta^2(T) + y^*(T)Ny(T)\right] + \delta_1\left[\frac{1}{2}\eta^2(0) - y^*(0)Ny(0)\right] + |C_1|,$$

with $0 < |C_1| < \infty$.

According to this, and $I_5 \leq C_1 \leq |C_1|$, $|C_1| = I_5 + \gamma$, $0 \leq \gamma < \infty$, we obtain the estimation (5.10). The theorem is proved. □

6 Global asymptotical stability

The following theorem gives a solution of Problem 1.2.

Theorem 6.1. *We assume that the following conditions hold:*

(i) *there is a vector* $\theta^* \in \mathbb{R}^n$ *such that* $\theta B = 0$, $\theta AB = 0, \dots, \theta A^{n-2} B = 0$, $k = \theta A^{n-1} B \neq 0$,

(ii) $\mu_1 \geq 0$,

(iii) *the rank of the matrix* $R = \|\theta^*, A^*\theta^*, \dots, A^{*n-1}\theta^*\|$ *is equal to* n,

(iv) $\int_\sigma^{\sigma+\Delta} \varphi(\xi) d\xi = \alpha$, $\alpha \neq 0$,

(v) $M_i < 0$, $i = \overline{1,n}$, *the matrix* $N \leq 0$.

Then the stationary set $\wedge$ *of the system* (1.1)–(1.2) *is globally asymptotically stable.*

Proof. Since the conditions of Theorem 5.4 hold, it follows that (5.10) is true. As the conditions (1.5) of the theorem hold, the improper integral satisfies $I_7 \geq 0$. Then, according to (5.10), functions $\eta(t)$, $y(t)$, $t \in I$, are bounded. If $\eta(t)$, $t \in I$, or $y(t)$, $t \in I$, is unbounded, then $0 \leq I_7 \leq -\infty$. This is impossible. Now, we obtain that (5.10) implies

$$
\begin{aligned}
0 \leq I_7 = \lim_{T\to\infty} \int_0^T \Big\{ & [\delta_1 + \delta_2(\tau_2 - \tau_1)] g(\eta(t), \sigma(t)) \eta(t) \\
& - \delta_1 [M_1 y_1^2(t) + \dots + M_n y_n^2(t)] \\
& + \frac{\delta_1^2 \nu^2}{4\delta_2(\tau_2\mu_2 - \tau_1\mu_1)} \varphi^2(\sigma(t)) \Big\} dt < \infty.
\end{aligned} \tag{6.1}
$$

From this we have that (see (6.1))

$$0 \leq \lim_{T\to\infty} \int_0^T [\delta_1 + \delta_2(\tau_2 - \tau_1)] g(\eta(t), \sigma(t)) \eta(t) < \infty, \tag{6.2}$$

$$0 \leq \lim_{T\to\infty} \int_0^T \delta_1 [M_1 y_1^2(t) + \dots + M_n y_n^2(t)] < \infty, \tag{6.3}$$

$$0 \leq \frac{\delta_1^2 \ \nu^2}{4\delta_2(\tau_2\mu_2 - \tau_1\mu_1)} \lim_{T\to\infty} \int_0^T \varphi^2(\sigma(t)) dt < \infty, \tag{6.4}$$

where $\delta_1 + \delta_2(\tau_2 - \tau_1) > 0$, $M_i < 0$, $i = \overline{1,n}$, $\tau_2\mu_2 - \tau_1\mu_1 > 0$, and the functions $\eta(t)$, $\sigma(t)$, $y_i(t)$, $i = \overline{1,n}$, $t \in I$, are uniformly continuous.

From (6.2)–(6.4), in the same way as in the proof of Theorem 4.1, we obtain $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} \eta(t) = 0$, $\lim_{t\to\infty} \sigma(t) = \sigma_*$, where $\varphi(\sigma_*) = 0$. It means that the stationary set $\wedge$ of the system (1.1)–(1.2) is globally asymptotically steady. The theorem is proved. □

There are processes described by equations of the following kind:

$$\frac{d\sigma}{dt} = \eta, \quad \frac{d\eta}{dt} = -g(\eta,\sigma) + x^* C \frac{d\overline{f}(\sigma)}{d\sigma} + \overline{\varphi}(\sigma), \tag{6.5}$$

$$\frac{dx}{dt} = Ax + \overline{B}\,\overline{f}(\sigma) + a, \tag{6.6}$$

where the functions $g(\eta,\sigma)$, $\overline{f}(\sigma)$, $\overline{\varphi}(\sigma)$ satisfy the conditions (1.3)–(1.4), in the theory of synchronous machines.

The system of the differential equations (6.5)–(6.6) by the nonsingular transformation (Lienar's transformation)

$$x(t) = z(t) - A^{-1}\overline{B}\,\overline{f}(\sigma) - A^{-1}a$$

can be reduced to the system of differential equations (1.1)–(1.2).

Indeed,

$$x^* C \frac{d\overline{f}(\sigma)}{d\sigma} = z^* C \frac{d\overline{f}(\sigma)}{d\sigma} - \overline{f}(\sigma)\overline{B}^* A^{*-1} C \frac{d\overline{f}(\sigma)}{d\sigma} - a^* A^* C \frac{d\overline{f}(\sigma)}{d\sigma}$$

and

$$\frac{dx}{dt} = \frac{dz}{dt} - A^{-1}\overline{B}\frac{d\overline{f}(\sigma)}{d\sigma}\eta = A[z - A^{-1}\overline{B}\,\overline{f}(\sigma) - A^{-1}a] + \overline{B}\,\overline{f}(\sigma) + a.$$

According to this, we can obtain that

$$\frac{d\sigma}{dt} = \eta, \quad \frac{d\eta}{dt} = -g(\eta,\sigma) + z^* C f(\sigma) + \varphi(\sigma),$$

where $B = A^{-1}\overline{B}$, $f(\sigma) = \frac{d\overline{f}(\sigma)}{d\sigma}$, $\varphi(\sigma) = \overline{\varphi}(\sigma) + [\frac{d\overline{f}(\sigma)}{d\sigma}]C^* A^{-1}[\overline{B}\,\overline{f}(\sigma) + a]$.

Various problems for synchronous machines can be found in paper [6].

7 Conclusion

A new method of investigation of global asymptotical stability for phase systems with countable set of equilibrium position is developed. By nonsingular transformation, the equations are reduced to a special form, and the properties of solutions are investigated.

Estimations of improper integrals along solutions of the system are obtained.

It is important that in evaluations the square forms are presented in the diagonal form such that allow to suggest a constructive method to allocate the region of attraction of global asymptotical stability in the space of constructive parameters of the system.

Bibliography

[1] P. M. Anderson and A. A. Fouad, *Power System Control and Stability*, The Iowa State University Press, Ames, 1977.

[2] S. A. Aisagaliev and B. K. Abenov, To global asymptotical stability of dynamical systemswith a cylindrical phase space (in Russian), *Vestnik KazNU, Ser. Math. Mech. Inf.* **2**(65) (2010), 7–13.

[3] S. A. Aisagaliev and S. A. Aipanov, *To the Theory of Global Asymptotical Stability of Phase Systems* (in Russian), Moskva, Minsk, 1999.

[4] S. A. Aisagaliev and T. S. Imankul, *Theory of Phase Systems* (in Russian), Kazakh University, Almaty, 2005.

[5] S. A. Aisagaliev and M. N. Kalimoldayev, *Theory of Dynamical Systems with a Cylindrical Phase Space* (in Russian), Gylym, Almaty, 2000.

[6] I. I. Blekhman, *Synchronization of Dynamical Systems* (in Russian), Nauka, Moscow, 1971.

[7] A. H. Gelig, G. A. Leonov and V. A. Yakubovich, *Stability of Non-Linear Systems with Non-Unique State of an Equilibrium Position* (in Russian), Nauka, Moscow, 1978.

[8] E. W. Kimbark, *Power System Stability. Volume I: Elements of Stability Calculations*, IEEE Press, New York, 1995.

[9] A. A. Yanko-Trinitskiy, *A New Method of the Analysis of Work of Synchronous Engines at Abrupt-Variableloadings* (in Russian), Gosenergoizdat, Moscow, 1958.

Received April 4, 2012.

Author information

Serikbai A. Aisagaliev, Almaty, Pushkin Street, 125, Kazakhstan.
E-mail: Serikbai.Aisagaliev@kaznu.kz

Maxat N. Kalimoldayev, Almaty, Pushkin Street, 125, Kazakhstan.
E-mail: mnk@ipic.kz